Advanced Mathematical Models & Applications
Vol.6, No.3, 2021, pp.278-291

ON RAINBOW ANTIMAGIC COLORING OF GRAPHS

Dafik[1*], Faisal Susanto[2], Ridho Alfarisi[1], Brian Juned Septory[1], Ika Hesti Agustin[2], M. Venkatachalam[3]

[1]CGANT - University of Jember, Jember, Indonesia
[2]Mathematics Department, University of Jember, Jember, Indonesia
[3]Mathematics Department, Kongunadu Arts and Science College, Tamil Nadu, India

Abstract. Let $G(V(G), E(G))$ be a connected, undirected, and simple graph with vertex set $V(G)$ and edge set $E(G)$. For a bijective function $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$, the associated weight of an edge $uv \in E(G)$ under f is $w_f(uv) = f(u) + f(v)$. The function f is called an edge-antimagic vertex labeling if every edge has distinct weight. A path P in the vertex-labeled graph G is said to be a rainbow path if for every two edges $uv, u'v' \in E(P)$ it satisfies $w_f(uv) \neq w_f(u'v')$. If for every two vertices u and v of G, there exists a rainbow $u - v$ path, then f is called a rainbow antimagic labeling of G. When we assign each edge uv with the color of the edge weight $w_f(uv)$, thus we say the graph G admits a rainbow antimagic coloring. The rainbow antimagic connection number of G, denoted by $rac(G)$, is the smallest number of colors taken over all rainbow colorings induced by rainbow antimagic labelings of G. To determine the rainbow antimagic connection number for any graph is considered to be a hard problem, even it is considered to be NP-Problem. In this paper, we will determine the rainbow antimagic connection number of graphs and characterize the lower and upper bound of the $rac(G)$ of graphs.

Keywords: Antimagic labeling, rainbow coloring, rainbow antimagic coloring, rainbow antimagic connection number.

AMS Subject Classification: 05C15.

Corresponding author: Dafik, CGANT - University of Jember, Jl. Kalimantan Tegalboto No.37, Jember, Indonesia, e-mail: *d.dafik@unej.ac.id*

Received: 28 September 2021; Revised: 23 November 2021; Accepted: 30 November 2021; Published: 27 December 2021.

1 Introduction

All graphs considered in this paper are finite, undirected, connected with neither loops nor multiple edges. For basic terminologies and notations of graphs, we follow Chartrand & Zhang (2016). Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of G is $|V(G)| = p$ and the size of G is $|E(G)| = q$. A graph labeling is one of big concepts in graph theory that has attracted many mathematicians around the globe. A graph labeling is a mapping from the set of elements in a graph (vertices, edges, or both) to the set of numbers (usually positive integers), called labels. There are many types of graph labeling techniques that have been established (see Gallian (2020) for the most complete survey on labelings).

In Hartsfield & Ringel (1990) defined antimagic graphs. A graph G is called antimagic if there exists a bijection $f : E(G) \to \{1, 2, \ldots, q\}$ such that the weights of all vertices are distinct. The vertex weight of a vertex v under f, $w_f(v)$, is the sum of labels of edges incident with v, that is, $w_f(v) = \sum_{uv \in E(G)} f(uv)$. In this case, f is called an antimagic labeling. Futhermore, an (a, d)-edge-antimagic vertex labeling of graphs was defined in Simanjuntak et al. (2000). For a given graph G with p vertices and q edges, a bijection $f : V(G) \to \{1, 2, 3, \ldots, p\}$ is called an (a, d)-edge-antimagic vertex labeling of G if the set of edge weights consists of an arithmetic progression $\{a, a+d, \ldots, a+(q-1)d\}$, where a and d are two fixed positive integers. The edge weight of an edge $e = uv$ under f, $w_f(uv)$, is the sum of labels of its end vertices, that is, $w_f(uv) = f(u) + f(v)$. A graph that admits an (a, d)-edge-antimagic vertex labeling is called an (a, d)-edge-antimagic vertex graph.

Another important topic in graph theory is graph colorings. In Chartrand et al. (2008) introduced the term rainbow coloring of a graph. Let $c : E(G) \to \{1, 2, \ldots, k\}$ be an edge k-coloring of a graph G where adjacent edges may be colored the same. A path in G is rainbow if no two edges of it are colored the same. The edge-colored graph G is rainbow-connected if every two distinct vertices are connected by a rainbow path. The edge k-coloring in which G is rainbow-connected is called a rainbow k-coloring. The minimum integer k in order to make G rainbow-connected is called the rainbow connection number of G and is denoted by $rc(G)$. This graph invariant has gained much attentions in Caro et al. (2008); Kemnitz & Schiermeyer (2011); Krivelevich & Yuster (2010); Li et al. (2012, 2017); Li & Shi (2013). The most complete survey on rainbow colorings can be found the survey by Li & Sun (2017).

In Chartrand et al. (2008) authors gave the following results for the rainbow connection number of cycles and trees.

Proposition 1. *Chartrand et al. (2008) The rainbow connection number of a cycle on n vertices is* $rc(C_n) = \lceil \frac{n}{2} \rceil$.

Proposition 2. *Chartrand et al. (2008) The rainbow connection number of a tree on n vertices is* $rc(T_n) = n - 1$.

The following prepositions and theorems are used to complete the proof of determining the antimagic rainbow connection number.

Proposition 3. *Septory et al. (2021) For any connected graph G, then* $rac(G) \geq rc(G)$.

In the following theorem, we give the general lower bound of rainbow antimagic connection number for any connected graph in terms of rainbow connection number and maximum degree of vertices of the graph.

Lemma 1. *Septory et al. (2021) Let G be any connected graph. Let* $rc(G)$ *and* $\Delta(G)$ *be the rainbow connection number of G and the maximum degree of G, respectively. Then* $rac(G) \geq \max\{rc(G), \Delta(G)\}$.

In this paper, we study the combination of two notions, namely antimagic labeling and rainbow coloring of graphs. This new concept has been initially studied in Septory et al. (2021); Sulistiyono et al. (2020); Budi et al. (2020); Jabbar et al. (2020) and obtained some rainbow antimagic connection on some basic graphs. We will continue to study the rainbow antimagic coloring and obtain the rainbow antimagic connection number of some graphs. We also analyse the lower bound of rainbow antimagic connection number for any connected graph. The characterizations of any graphs having the rainbow antimagic coloring are also studied in this paper.

2 Results

Prior to show our new results, we will show the following propositions.

Proposition 4. *If G is an edge antimagic graph, then G admits a rainbow antimagic coloring.*

This basic fact naturally hold for an edge antimagic graph, since the evaluation of edge antimagic labeling is done in each edge of G, and all edge weights are different. It implies that there must exist a rainbow path between any two vertices. Furthermore, the following theorem shows the existence of a rainbow $u - v$ path for any two vertices u and v with distance at most two.

Theorem 1. *Let G be a connected graph of diameter $diam(G) \leq 2$. Let f be any bijective function from $V(G)$ to the set $\{1, 2, \ldots, |V(G)|\}$. For $u, v \in V(G)$, there exists a rainbow path $u - v$.*

Proof. Let $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijection, and f admits edge antimagic laeling. Clearly, if $diam(G) = 1$ then the maksimum $d(u, v) = 1$. There must exist a rainbow $u - v$ path, namely the edge uv. When $diam(G) = 2$, we consider two vertices u and v of G with $d(u, v) \leq 2$. Let z be a vertex in $V(G) \backslash \{u, v\}$ such that z is adjacent to u and v. Since f is bijective, it must be $w_f(uz) \neq w_f(vz)$. Consequently, there exists a rainbow $u - v$ path, namely uzv. □

From Theorem 1 we immediately get the following theorem.

Corollary 1. *Let G be any connected graph of order at least two and diameter at most two. Then, G admits a rainbow antimagic coloring.*

Theorem 2. *For $\forall n \geq 2$ where $n \in \mathcal{N}$, $rac(K_n) = 2n - 3$.*

Proof. The complete graph on n vertices K_n is a graph in which every pair of distinct vertices is connected by an edge. The graph K_n is rainbow antimagic according to Corollary 1. Trivially, in any rainbow antimagic coloring of K_n, the smallest weight of edges must be $1 + 2 = 3$, the largest weight of edges must be $n - 1 + n = 2n - 1$ and the set of distinct edge weights is $\{3, 4, 5, 6, \ldots, 2n - 1\}$ which gives $2n - 1 - 3 + 1 = 2n - 3$ elements. It is the number of colors required to color G such that it admits a rainbow antimagic coloring. Hence, we get the following result. □

In Figure 1(a), we give an illustration of the rainbow antimagic coloring of K_6. Furthermore, in the next theorem, we will prove that the cycle with n vertices C_n admits a rainbow antimagic coloring. We describe its rainbow antimagic connection number in the following.

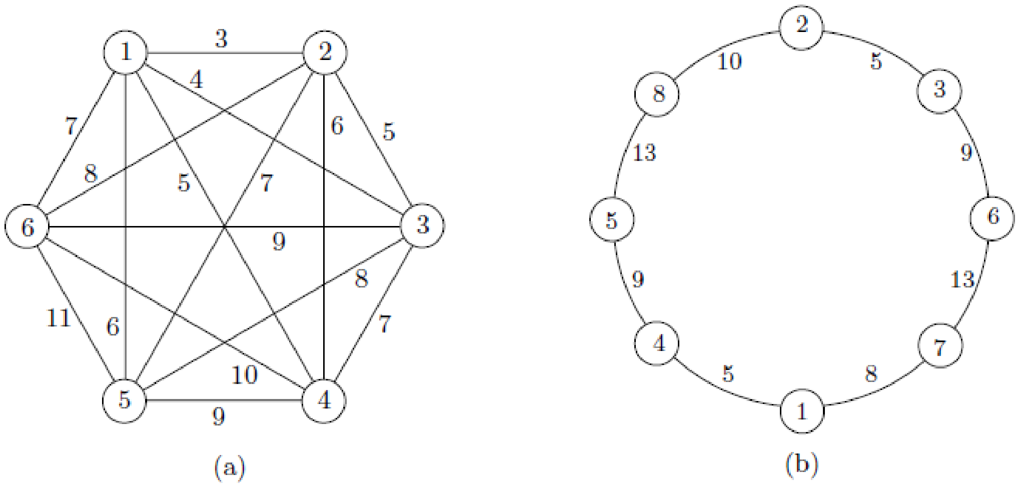

Figure 1: (a) A rainbow antimagic coloring of K_6 (b) A rainbow antimagic coloring of C_8

Theorem 3. *For $\forall n \geq 3$ where $n \in \mathcal{N}$, then*

$$rac(C_n) = \begin{cases} 3, & \text{if } n = 4, \\ \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 1, 2 \pmod 4, \end{cases}$$

and

$$\left\lceil \frac{n}{2} \right\rceil \leq rac(C_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1, \quad \text{if } n \equiv 0, 3 \pmod 4), n \neq 4.$$

Proof. Let C_n be a cycle with vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$. Let us consider three cases.

Case 1. If $n = 4$. By Proposition 1 and Lemma 1, then $rac(C_4) \geq 2$. Assume, to the contrary, $rac(C_4) = 2$. Let $f : \{v_1, v_2, v_3, v_4\} \to \{1, 2, 3, 4\}$ be a rainbow antimagic coloring of C_4 such that f induces a rainbow 2-coloring. It is obvious that $w_f(v_1v_2) = w_f(v_3v_4)$ and $w_f(v_2v_3) = w_f(v_4v_1)$. Therefore, we have $f(v_1)+f(v_2) = f(v_3)+f(v_4)$ and $f(v_2)+f(v_3) = f(v_4)+f(v_1)$, which lead to $f(v_1) = f(v_3)$, a contradiction. Thus, $rac(C_4) \geq 3$. A rainbow 3-coloring induced by a rainbow antimagic coloring of C_4 is shown in Figure 2. Hence, $rac(C_4) \leq 3$. Combining with the lower bound, we have $rac(C_4) = 3$.

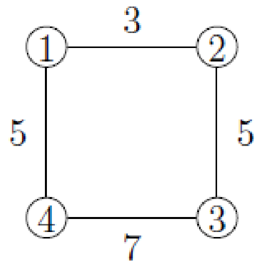

Figure 2: A rainbow 3-coloring induced by a rainbow antimagic coloring of C_4.

Case 2. If $n \equiv 1, 2 \pmod 4$. According to Proposition 1 and Theorem 1, $rac(C_n) \geq \lceil n/2 \rceil$. To realize the equality, let us define a vertex labeling $f : V(C_n) \to \{1, 2, 3, \ldots, n\}$ such that

$$
\begin{array}{ll}
f(v_i) = 2i, & \text{if } n \equiv 1 \pmod 4, \text{ for } i = 1, 3, \ldots, \frac{n-3}{2}, \text{ or} \\
 & \text{if } n \equiv 2 \pmod 4, \text{ for } i = 1, 3, \ldots, \frac{n}{2}, \\
f(v_i) = 2i - 1, & \text{if } n \equiv 1 \pmod 4, \text{ for } i = 2, 4, \ldots, \frac{n-1}{2}, \text{ or} \\
 & \text{if } n \equiv 2 \pmod 4, \text{ for } i = 2, 4, \ldots, \frac{n-2}{2}, \\
f(v_{\frac{n+1}{2}}) = n, & \text{if } n \equiv 1 \pmod 4, \\
f(v_i) = 2i - n - 2, & \text{if } n \equiv 1 \pmod 4, \text{ for } i = \frac{n+3}{2}, \frac{n+7}{2}, \ldots, n-1, \\
f(v_i) = 2i - n - 1, & \text{if } n \equiv 1 \pmod 4, \text{ for } i = \frac{n+5}{2}, \frac{n+9}{2}, \ldots, n, \text{ or} \\
 & \text{if } n \equiv 2 \pmod 4, \text{ for } i = \frac{n+2}{2}, \frac{n+6}{2}, \ldots, n, \\
f(v_i) = 2i - n, & \text{if } n \equiv 2 \pmod 4, \text{ for } i = \frac{n+4}{2}, \frac{n+8}{2}, \ldots, n-1.
\end{array}
$$

Then, the labeling f provides the vertex weights as follows.

$$
\begin{array}{ll}
w_f(v_iv_{i+1}) = 4i + 1, & \text{if } n \equiv 1 \pmod 4, \text{ for } i = 1, 2, \ldots, \frac{n-3}{2}, \text{ or} \\
 & \text{if } n \equiv 2 \pmod 4, \text{ for } i = 1, 2, \ldots, \frac{n-2}{2}, \\
w_f(v_{\frac{n-1}{2}}v_{\frac{n+1}{2}}) = 2n - 2, & \text{if } n \equiv 1 \pmod 4,
\end{array}
$$

$$
\begin{aligned}
&w_f(v_iv_{i+1}) = n+1, && \text{if } n = 1 \pmod 4, \text{ for } i = \frac{n+1}{2}, \text{ or}\\
& && \text{if } n = 2 \pmod 4, \text{ for } i = \frac{n}{2}, \text{ or}\\
&w_f(v_iv_{i+1}) = 4i-2n-1, && \text{if } n \equiv 1 \pmod 4, \text{ for } i = \frac{n+3}{2}, \frac{n+5}{2}, \dots, n-1,\\
&w_f(v_iv_{i+1}) = 4i-2n+1, && \text{if } n \equiv 2 \pmod 4, \text{ for } i = \frac{n+2}{2}, \frac{n+4}{2}, \dots, n-1,\\
&w_f(v_nv_1) = n+1, && \text{if } n \equiv 1,2 \pmod 4.
\end{aligned}
$$

Thus, if $n \equiv 1 \pmod 4$, we get

$$
\begin{aligned}
\{wt_f(e) : e \in E(C_n)\} = &\left\{4i+1 : i = 1,2,\dots,\frac{n-3}{2}\right\} \cup \{2n-2\} \cup \{n+1\}\\
&\cup \left\{4i-2n-1 : i = \frac{n+3}{2}, \frac{n+5}{2}, \dots, n-1\right\} \cup \{n+1\}\\
=&\{5,9,\dots,2n-5\} \cup \{2n-2\} \cup \{n+1\} \cup \{5,9,\dots,2n-5\} \cup \{n+1\}\\
=&\{5,9,\dots,2n-5,2n-2,n+1\}
\end{aligned}
$$

and if $n \equiv 2 \pmod 4$, we have

$$
\begin{aligned}
\{wt_f(e) : e \in E(C_n)\} = &\left\{4i+1 : i = 1,2,\dots,\frac{n-2}{2}\right\} \cup \{n+1\}\\
&\cup \left\{4i-2n+1 : i = \frac{n+2}{2}, \frac{n+4}{2}, \dots, n-1\right\} \cup \{n+1\}\\
=&\{5,9,\dots,2n-3\} \cup \{n+1\} \cup \{5,9,\dots,2n-3\} \cup \{n+1\}\\
=&\{5,9,\dots,2n-3,n+1\}.
\end{aligned}
$$

Therefore,

$$
|\{wt_f(e) : e \in E(C_n)\}| = \begin{cases} \frac{n+1}{2} = \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 1 \pmod 4,\\ \frac{n}{2} = \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 2 \pmod 4. \end{cases}
$$

It can be seen that f induces an edge $\lceil n/2 \rceil$-coloring of C_n. Further, it is not difficult to show that this coloring is a rainbow coloring of C_n. Hence, $rac(C_n) \le \lceil n/2 \rceil$. We can conclude that $rac(C_n) = \lceil \frac{n}{2} \rceil$.

Case 3. If $n \equiv 0,3 \pmod 4$. From Proposition 1 and Theorem 1, we have $rac(C_n) \ge \lceil n/2 \rceil$. Next, define a vertex labeling $f : V(C_n) \to \{1,2,\dots,n\}$ as follows.

$$
\begin{aligned}
&f(v_i) = 2i, && \text{if } n \equiv 0 \pmod 4, \text{ for } i = 1,3,\dots,\frac{n-2}{2}, \text{ or}\\
& && \text{if } n \equiv 3 \pmod 4, \text{ for } i = 1,3,\dots,\frac{n-1}{2},\\
&f(v_i) = 2i-1, && \text{if } n \equiv 0 \pmod 4, \text{ for } i = 2,4,\dots,\frac{n}{2}, \text{ or}\\
& && \text{if } n \equiv 3 \pmod 4, \text{ for } i = 2,4,\dots,\frac{n+1}{2},\\
&f(v_i) = 2i-n-1, && \text{if } n \equiv 0 \pmod 4, \text{ for } i = \frac{n+2}{2}, \frac{n+6}{2}, \dots, n-1, \text{ or}\\
& && \text{if } n \equiv 3 \pmod 4, \text{ for } i = \frac{n+5}{2}, \frac{n+9}{2}, \dots, n-1,\\
&f(v_i) = 2i-n-2, && \text{if } n \equiv 3 \pmod 4, \text{ for } i = \frac{n+3}{2}, \frac{n+7}{2}, \dots, n,\\
&f(v_i) = 2i-n, && \text{if } n \equiv 0 \pmod 4, \text{ for } i = \frac{n+4}{2}, \frac{n+8}{2}, \dots, n.
\end{aligned}
$$

For the vertex weights, we have

$$\begin{aligned}
w_f(v_iv_{i+1}) &= 4i+1, && \text{if } n \equiv 0 \pmod 4,\ \text{for } i = 1,2,\ldots,\frac{n-2}{2},\ \text{or}\\
& && \text{if } n \equiv 3 \pmod 4,\ \text{for } i = 1,2,\ldots,\frac{n-1}{2},\\
w_f(v_{\frac{n}{2}}v_{\frac{n+2}{2}}) &= n, && \text{if } n \equiv 0 \pmod 4,\\
w_f(v_{\frac{n+1}{2}}v_{\frac{n+3}{2}}) &= n+1, && \text{if } n \equiv 3 \pmod 4,\\
w_f(v_iv_{i+1}) &= 4i-2n+1, && \text{if } n \equiv 0 \pmod 4,\ \text{for } i = \frac{n+2}{2},\frac{n+4}{2},\ldots,n-1,\\
w_f(v_iv_{i+1}) &= 4i-2n-1, && \text{if } n \equiv 3 \pmod 4,\ \text{for } i = \frac{n+3}{2},\frac{n+5}{2},\ldots,n-1,\\
w_f(v_nv_1) &= n+2, && \text{if } n \equiv 0 \pmod 4,\\
w_f(v_nv_1) &= n, && \text{if } n \equiv 3 \pmod 4.
\end{aligned}$$

It can be verified that f induces an edge $(\lceil n/2\rceil+1)$-coloring of C_n. Moreover, it can be checked that this coloring is a rainbow coloring. Hence, $rac(C_n) \le \lceil n/2\rceil + 1$. Combining with the lower bound, we obtain $\lceil n/2\rceil \le rac(C_n) \le \lceil n/2\rceil + 1$. □

For example, we give in Figure 1(b) a rainbow antimagic coloring of C_8 as an illustration.

Next, we study the rainbow antimagic connection number of a fan graph. The fan graph, F_n, is a graph obtained from a path P_n by adding a vertex and joining it to all n vertices of P_n.

Theorem 4. *For $\forall n \ge 3$ where $n \in \mathcal{N}$, then*

$$rac(F_n) = \begin{cases} 4, & \text{if } n = 3,\\ n, & \text{if } n \ge 4. \end{cases}$$

Proof. Let F_n be a fan with vertex set $V(F_n) = \{v\} \cup \{v_i : i = 1,2,\ldots,n\}$ and edge set $E(F_n) = \{vv_i : i = 1,2,\ldots,n\} \cup \{v_iv_{i+1} : i = 1,2,\ldots,n-1\}$. We separate our proof into two cases below.

Case 1. If $n = 3$. By Theorem 1, $rac(F_3) \ge 3$. However, we will show that $rac(F_3) \ge 4$. For a contradiction, suppose that $rac(F_3) = 3$. Let $f : \{v, v_1, v_2, v_3\} \to \{1,2,3,4\}$ be a rainbow antimagic coloring of F_3 such that f induces a rainbow 3-coloring. We know that $w_f(vv_1) = f(v) + f(v_1)$, $w_f(vv_2) = f(v) + f(v_2)$, $w_f(vv_3) = f(v) + f(v_3)$, $w_f(v_1v_2) = f(v_1) + f(v_2)$ and $w_f(v_2v_3) = f(v_2) + f(v_3)$. Clearly, $w_f(vv_1) \ne w_f(vv_2) \ne w_f(vv_3)$. Since v_1v_2 is adjacent to vv_1 and vv_2, then $w_f(v_1v_2) \ne w_f(vv_1)$ and $w_f(v_1v_2) \ne w_f(vv_2)$. Consequently, v_1v_2 must be colored with the same color as vv_3, that is,

$$w_f(v_1v_2) = w_f(vv_3) \text{ or } f(v_1) + f(v_2) = f(v) + f(v_3). \tag{1}$$

By the similar argument, v_2v_3 must be colored with the same color as vv_1, that is,

$$w_f(v_2v_3) = w_f(vv_1) \text{ or } f(v_2) + f(v_3) = f(v) + f(v_1). \tag{2}$$

From (1) and (2), we get $f(v_1) = f(v_3)$, contradicts to the assumption that f is a rainbow antimagic coloring of F_3. So, $rac(F_3) \ge 4$. Next, for the upper bound, label the vertices of F_3 as in Figure 3. Thus, $rac(F_3) \le 4$. Combining with the lower bound, then $rac(F_3) = 4$.

Figure 3: A rainbow 4-coloring induced by a rainbow antimagic coloring of F_3.

Case 2. If $n \geq 4$. By Theorem 1, we have $rac(F_n) \geq n$. To prove the upper bound, let us define a vertex labeling $f(V(F_n)) \to \{1, 2, \dots, n+1\}$ as follows.
For $n = 4, 5$, label the vertices of F_n by using the following formula.

$$\begin{aligned}
f(v) &= n-2, && \\
f(v_{n-4}) &= 1, && \\
f(v_i) &= 2n-4-i, && \text{for } i = n-3, n-1, \\
f(v_i) &= 1 + \frac{n+i}{2}, && \text{for } i = n-2, n.
\end{aligned}$$

Therefore, the vertex weights are

$$\begin{aligned}
w_f(vv_{n-4}) &= n-1, && \\
w_f(vv_i) &= 3n-6-i, && \text{for } i = n-3, n-1, \\
w_f(vv_i) &= n-1+\frac{n+i}{2}, && \text{for } i = n-2, n, \\
w_f(v_{n-4}v_{n-3}) &= n, && \\
w_f(v_{n-3}v_{n-2}) &= 2n-1, && \\
w_f(v_{n-2}v_{n-1}) &= 2n-3, && \\
w_f(v_{n-1}v_n) &= 2n-2. &&
\end{aligned}$$

For $n \geq 6$, label the vertices of F_n as follows.

$$\begin{aligned}
f(v) &= \left\lfloor \frac{n}{2} \right\rfloor, && \\
f(v_i) &= i, && \text{if } n \equiv 0 \pmod 4, \text{ for } i = 1, 3, \dots, \lfloor \tfrac{n+2}{2} \rfloor, \text{ or} \\
& && \text{if } n \equiv 1 \pmod 4, \text{ for } i = 1, 3, \dots, \lfloor \tfrac{n-2}{2} \rfloor, \text{ or} \\
& && \text{if } n \equiv 2, 3 \pmod 4, \text{ for } i = 1, 3, \dots, \lfloor \tfrac{n-4}{2} \rfloor, \\
f(v_i) &= n+2-i, && \text{if } n \equiv 0 \pmod 4, \text{ for } i = 2, 4, \dots, \lfloor \tfrac{n}{2} \rfloor, \text{ or} \\
& && \text{if } n \equiv 1 \pmod 4, \text{ for } i = 2, 4, \dots, \lfloor \tfrac{n+4}{2} \rfloor, \text{ or} \\
& && \text{if } n \equiv 2 \pmod 4, \text{ for } i = 2, 4, \dots, \lfloor \tfrac{n-2}{2} \rfloor, \text{ or} \\
& && \text{if } n \equiv 3 \pmod 4, \text{ for } i = 2, 4, \dots, \lfloor \tfrac{n+2}{2} \rfloor, \\
f(v_i) &= n+1-i, && \text{if } n \equiv 0, 3 \pmod 4, \text{ for } i = \lfloor \tfrac{n+6}{2} \rfloor, \lfloor \tfrac{n+10}{2} \rfloor, \dots, n-1, \text{ or} \\
& && \text{if } n \equiv 1 \pmod 4, \text{ for } i = \lfloor \tfrac{n+8}{2} \rfloor, \lfloor \tfrac{n+12}{2} \rfloor, \dots, n-1, \text{ or} \\
& && \text{if } n \equiv 2 \pmod 4, \text{ for } i = \lfloor \tfrac{n}{2} \rfloor, \lfloor \tfrac{n+4}{2} \rfloor, \dots, n-1,
\end{aligned}$$

$$\begin{aligned}
f(v_i) &= i+1, && \text{if } n \equiv 0 \pmod 4, \text{ for } i = \lfloor \tfrac{n+4}{2} \rfloor, \lfloor \tfrac{n+8}{2} \rfloor, \dots, n, \text{ or} \\
& && \text{if } n \equiv 1, 2 \pmod 4, \text{ for } i = \lfloor \tfrac{n+2}{2} \rfloor, \lfloor \tfrac{n+6}{2} \rfloor, \dots, n, \text{ or} \\
& && \text{if } n \equiv 3 \pmod 4, \text{ for } i = \lfloor \tfrac{n}{2} \rfloor, \lfloor \tfrac{n+4}{2} \rfloor, \dots, n.
\end{aligned}$$

The labeling f gives the vertex weights below.

$$
\begin{array}{ll}
w_f(vv_i) = \lfloor \frac{n}{2} \rfloor + i, & \text{if } n \equiv 0 \pmod 4, \text{ for } i = 1, 3, \ldots, \lfloor \frac{n+2}{2} \rfloor, \text{ or} \\
 & \text{if } n \equiv 1 \pmod 4, \text{ for } i = 1, 3, \ldots, \lfloor \frac{n-2}{2} \rfloor, \text{ or} \\
 & \text{if } n \equiv 2, 3 \pmod 4, \text{ for } i = 1, 3, \ldots, \lfloor \frac{n-4}{2} \rfloor, \\
w_f(vv_i) = \lfloor \frac{3n}{2} \rfloor + 2 - i, & \text{if } n \equiv 0 \pmod 4, \text{ for } i = 2, 4, \ldots, \lfloor \frac{n}{2} \rfloor, \text{ or} \\
 & \text{if } n \equiv 1 \pmod 4, \text{ for } i = 2, 4, \ldots, \lfloor \frac{n+4}{2} \rfloor, \text{ or} \\
 & \text{if } n \equiv 2 \pmod 4, \text{ for } i = 2, 4, \ldots, \lfloor \frac{n-2}{2} \rfloor, \text{ or} \\
 & \text{if } n \equiv 3 \pmod 4, \text{ for } i = 2, 4, \ldots, \lfloor \frac{n+2}{2} \rfloor, \\
w_f(vv_i) = \lfloor \frac{3n}{2} \rfloor + 1 - i, & \text{if } n \equiv 0, 3 \pmod 4, \text{ for } i = \lfloor \frac{n+6}{2} \rfloor, \lfloor \frac{n+10}{2} \rfloor, \ldots, n-1, \text{ or} \\
 & \text{if } n \equiv 1 \pmod 4, \text{ for } i = \lfloor \frac{n+8}{2} \rfloor, \lfloor \frac{n+12}{2} \rfloor, \ldots, n-1, \text{ or} \\
 & \text{if } n \equiv 2 \pmod 4, \text{ for } i = \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n+4}{2} \rfloor, \ldots, n-1, \\
w_f(vv_i) = \lfloor \frac{n}{2} \rfloor + 1 + i, & \text{if } n \equiv 0 \pmod 4, \text{ for } i = \lfloor \frac{n+4}{2} \rfloor, \lfloor \frac{n+8}{2} \rfloor, \ldots, n, \text{ or} \\
 & \text{if } n \equiv 1, 2 \pmod 4, \text{ for } i = \lfloor \frac{n+2}{2} \rfloor, \lfloor \frac{n+6}{2} \rfloor, \ldots, n, \text{ or} \\
 & \text{if } n \equiv 3 \pmod 4, \text{ for } i = \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n+4}{2} \rfloor, \ldots, n, \\
w_f(v_iv_{i+1}) = n + 1, & \text{if } n \equiv 0 \pmod 4, \text{ for } i = 1, 3, \ldots, \lfloor \frac{n-2}{2} \rfloor \text{ and} \\
 & \quad i = \lfloor \frac{n+4}{2} \rfloor, \lfloor \frac{n+8}{2} \rfloor, \ldots, n-2, \text{ or} \\
 & \text{if } n \equiv 1 \pmod 4, \text{ for } i = 1, 3, \ldots, \lfloor \frac{n-2}{2} \rfloor \text{ and} \\
 & \quad i = \lfloor \frac{n+6}{2} \rfloor, \lfloor \frac{n+10}{2} \rfloor, \ldots, n-2, \text{ or} \\
 & \text{if } n \equiv 2 \pmod 4, \text{ for } i = 1, 3, \ldots, \lfloor \frac{n-4}{2} \rfloor, \text{ or} \\
 & \text{if } n \equiv 3 \pmod 4, \text{ for } i = 1, 3, \ldots, \lfloor \frac{n-4}{2} \rfloor \text{ and} \\
 & \quad i = \lfloor \frac{n+4}{2} \rfloor, \lfloor \frac{n+8}{2} \rfloor, \ldots, n-2, \\
w_f(v_iv_{i+1}) = n + 2, & \text{if } n \equiv 1 \pmod 4, \text{ for } i = \lfloor \frac{n+2}{2} \rfloor, \text{ or} \\
 & \text{if } n \equiv 3 \pmod 4, \text{ for } i = \lfloor \frac{n}{2} \rfloor,
\end{array}
$$

$$w_f(v_iv_{i+1}) = n+3, \qquad \text{if } n \equiv 0 \pmod 4,\ \text{for } i = 2,4,\ldots,\lfloor\frac{n}{2}\rfloor \text{ and}$$
$$i = \lfloor\frac{n+6}{2}\rfloor, \lfloor\frac{n+10}{2}\rfloor, \ldots, n-1,\ \text{or}$$
$$\text{if } n \equiv 1 \pmod 4,\ \text{for } i = 2,4,\ldots,\lfloor\frac{n-4}{2}\rfloor \text{ and}$$
$$i = \lfloor\frac{n+8}{2}\rfloor, \lfloor\frac{n+12}{2}\rfloor, \ldots, n-1,\ \text{or}$$
$$\text{if } n \equiv 2 \pmod 4,\ \text{for } i = \lfloor\frac{n}{2}\rfloor, \lfloor\frac{n+4}{2}\rfloor, \ldots, n-1,\ \text{or}$$
$$\text{if } n \equiv 3 \pmod 4,\ \text{for } i = 2,4,\ldots,\lfloor\frac{n-6}{2}\rfloor \text{ and}$$
$$i = \lfloor\frac{n+6}{2}\rfloor, \lfloor\frac{n+10}{2}\rfloor, \ldots, n-1,$$
$$w_f(v_iv_{i+1}) = n+4, \qquad \text{if } n \equiv 0 \pmod 4,\ \text{for } i = \lfloor\frac{n+2}{2}\rfloor,\ \text{or}$$
$$\text{if } n \equiv 1 \pmod 4,\ \text{for } i = \lfloor\frac{n}{2}\rfloor, \lfloor\frac{n+4}{2}\rfloor,\ \text{or}$$
$$\text{if } n \equiv 2 \pmod 4,\ \text{for } i = 2,4,\ldots,\lfloor\frac{n-2}{2}\rfloor,\ \text{or}$$
$$\text{if } n \equiv 3 \pmod 4,\ \text{for } i = \lfloor\frac{n-2}{2}\rfloor, \lfloor\frac{n+2}{2}\rfloor.$$

It can be checked that f induces an edge n-coloring. Since $diam(F_n) = 2$, from Lemma 1, then there exists a rainbow $x-y$ path for any two vertices $x, y \in V(F_n)$. Therefore, we can conclude that $rac(F_n) \le n$. It completes the proof. □

For an illustration, in Figure 5(a), a rainbow antimagic coloring of F_8 is given.

A wheel, W_n, is a graph obtained by joining a vertex to n vertices of a cycle. In the next theorem, we determine the rainbow antimagic connection number of W_n.

Theorem 5. *For $\forall n \ge 3$ where $n \in \mathcal{N}$, then*

$$rac(W_n) = \begin{cases} 5, & \text{if } n = 3,4, \\ n, & \text{if } n \ge 5. \end{cases}$$

Proof. Let W_n be a wheel with vertex set $V(W_n) = \{v\} \cup \{v_i : i = 1,2,\ldots,n\}$ and edge set $E(W_n) = \{vv_i, v_iv_{i+1} : i = 1,2,\ldots,n\}$, where the index i is taken modulo n. We divide the proof into two cases as follows.

Case 1. If $n = 3,4$. For $n = 3$, $W_3 \cong K_4$. By Theorem 2, $rac(W_3) = 5$. Now, consider the case for $n = 4$. From Lemma 1, we have $rac(W_4) \ge 4$. However, it does not attain the best lower bound, instead we have $rac(W_4) \ge 5$. For a contradiction, suppose that $rac(W_4) = 4$. Let $f : \{v, v_1, v_2, v_3, v_4\} \to \{1,2,3,4,5\}$ be a rainbow antimagic labeling of W_4 such that f induces a rainbow 4-coloring. We know that $w_f(vv_1) = f(v)+f(v_1)$, $w_f(vv_2) = f(v)+f(v_2)$, $w_f(vv_3) = f(v) + f(v_3)$, $w_f(vv_4) = f(v) + f(v_4)$, $w_f(v_1v_2) = f(v_1) + f(v_2)$, $w_f(v_2v_3) = f(v_2) + f(v_3)$, $w_f(v_3v_4) = f(v_3)+f(v_4)$ and $w_f(v_4v_1) = f(v_4)+f(v_1)$. Obviously, we have $w_f(vv_1) \ne w_f(vv_2) \ne w_f(vv_3) \ne w_f(vv_4)$. For the color of other edges, we have the following subcases.

Subcase 1.1. If $w_f(v_1v_2) = w_f(vv_3)$, then, respectively, we have $w_f(v_4v_1) = w_f(vv_2)$, $w_f(v_3v_4) = w_f(vv_1)$ and $w_f(v_2v_3) = w_f(vv_4)$. From the these facts, we get an equation system below.

$$\begin{aligned} f(v_1) + f(v_2) &= f(v) + f(v_3) \\ f(v_4) + f(v_1) &= f(v) + f(v_2) \\ f(v_3) + f(v_4) &= f(v) + f(v_1) \\ f(v_2) + f(v_3) &= f(v) + f(v_4) \end{aligned} \tag{3}$$

Solving (3) by using Gauss-Jordan elimination method, we can obtain $f(v) = f(v_1) = f(v_2) = f(v_3) = f(v_4)$, a contradiction.

Subcase 1.2. If $w_f(v_1v_2) = w_f(vv_4)$, then, respectively, we have $w_f(v_2v_3) = w_f(vv_1)$, $w_f(v_3v_4) = w_f(vv_2)$ and $w_f(v_4v_1) = w_f(vv_3)$. From these facts, we get an equation system below.

$$\begin{aligned} f(v_1) + f(v_2) &= f(v) + f(v_4) \\ f(v_2) + f(v_3) &= f(v) + f(v_1) \\ f(v_3) + f(v_4) &= f(v) + f(v_2) \\ f(v_4) + f(v_1) &= f(v) + f(v_3) \end{aligned} \tag{4}$$

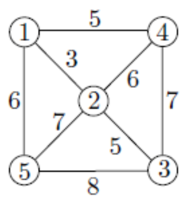

Figure 4: A rainbow 5-coloring induced by a rainbow antimagic labeling of W_4.

Solving (4) by using Gauss-Jordan elimination method, we can obtain $f(v) = f(v_1) = f(v_2) = f(v_3) = f(v_4)$, which again a contradiction. From all subcases, 4 colors are not enough to color the graph W_4. Therefore, $rac(W_4) \geq 5$. A rainbow 5-coloring induced by a rainbow antimagic labeling of W_4 is given in Figure 4. Hence, $rac(W_4) \leq 5$. Combining with the lower bound, we have $rc_{la}(W_4) = 5$.

Case 2. If $n \geq 5$. From Theorem 1, $rc_{la}(W_n) \geq n$. Let $f : V(W_n) \to \{1, 2, \ldots, n+1\}$ be a vertex labeling of W_n defined such that

$$\begin{aligned}
f(v) &= \left\lceil \frac{n}{2} \right\rceil + 1, & & \\
f(v_i) &= \frac{i+1}{2}, & & \text{if } n \equiv 1 (\text{mod } 2),\ \text{for } i = 1, 3, \ldots, n-2,\ \text{or} \\
& & & \text{if } n \equiv 0 (\text{mod } 2),\ \text{for } i = 1, 3, \ldots, n-1, \\
f(v_i) &= n + 1 - \frac{i}{2}, & & \text{if } n \equiv 1 (\text{mod } 2),\ \text{for } i = 2, 4, \ldots, n-3, \\
f(v_{n-1}) &= \frac{n+1}{2}, & & \text{if } n \equiv 1 (\text{mod } 2), \\
f(v_n) &= n + 1, & & \text{if } n \equiv 1 (\text{mod } 2), \\
f(v_2) &= \frac{n}{2} + 2, & & \text{if } n \equiv 0 (\text{mod } 2), \\
f(v_i) &= n + 3 - \frac{i}{2}, & & \text{if } n \equiv 0 (\text{mod } 2),\ \text{for } i = 4, 6, \ldots, n.
\end{aligned}$$

Then, the labeling f yields the the following vertex weights.

$$\begin{aligned}
w_f(vv_i) &= \lceil \frac{n}{2} \rceil + \frac{i+3}{2}, & & \text{if } n \equiv 1 (\text{mod } 2),\ \text{for } i = 1, 3, \ldots, n-2,\ \text{or} \\
& & & \text{if } n \equiv 0 (\text{mod } 2),\ \text{for } i = 1, 3, \ldots, n-1, \\
w_f(vv_i) &= \frac{3n + 5 - i}{2}, & & \text{if } n \equiv 1 (\text{mod } 2),\ \text{for } i = 2, 4, \ldots, n-3, \\
w_f(vv_{n-1}) &= n + 2, & & \text{if } n \equiv 1 (\text{mod } 2), \\
w_f(vv_n) &= \frac{3n+5}{2}, & & \text{if } n \equiv 1 (\text{mod } 2),
\end{aligned}$$

$$
\begin{aligned}
w_f(vv_2) &= n+3, && \text{if } n \equiv 0 (\text{mod } 2),\\
w_f(vv_i) &= 4+\frac{3n-i}{2}, && \text{if } n \equiv 0 (\text{mod } 2),\ \text{for } i = 4,6,\dots,n,\\
w_f(v_iv_{i+1}) &= n+1, && \text{if } n \equiv 1 (\text{mod } 2),\ \text{for } i = 1,3,\dots,n-4,\\
w_f(v_iv_{i+1}) &= n+2, && \text{if} n \equiv 1 (\text{mod } 2),\ \text{for } i = 2,4,\dots,n-3,\\
w_f(v_{n-2}v_{n-1}) &= n, && \text{if } n \equiv 1 (\text{mod } 2),\\
w_f(v_{n-1}v_n) &= \frac{3n+3}{2}, && \text{if } n \equiv 1 (\text{mod } 2),\\
w_f(v_1v_2) &= \frac{n}{2}+3, && \text{if } n \equiv 0 (\text{mod } 2),\\
w_f(v_2v_3) &= \frac{n}{2}+4, && \text{if } n \equiv 0 (\text{mod } 2),\\
w_f(v_iv_{i+1}) &= n+3, && \text{if } n \equiv 0 (\text{mod } 2),\ \text{for } i = 3,5,\dots,n-1,\\
w_f(v_iv_{i+1}) &= n+4, && \text{if } n \equiv 0 (\text{mod } 2),\ \text{for } i = 4,6,\dots,n-2,\\
w_f(v_nv_1) &= n+2, && \text{if } n \equiv 1 (\text{mod } 2),\\
w_f(v_nv_1) &= \frac{n}{2}+4, && \text{if } n \equiv 0 (\text{mod } 2).
\end{aligned}
$$

We can see that f induces an edge n-coloring. Since $diam(W_n) = 2$, from Lemma 1, then there exists a rainbow $x-y$ path for any two vertices $x, y \in V(F_n)$. So, $rac(W_n) \leq n$. Combining with the lower bound, then $rac(W_n) = n$. □

For an illustration, in Figure 5(b), it is given an example of a rainbow antimagic coloring of W_6.

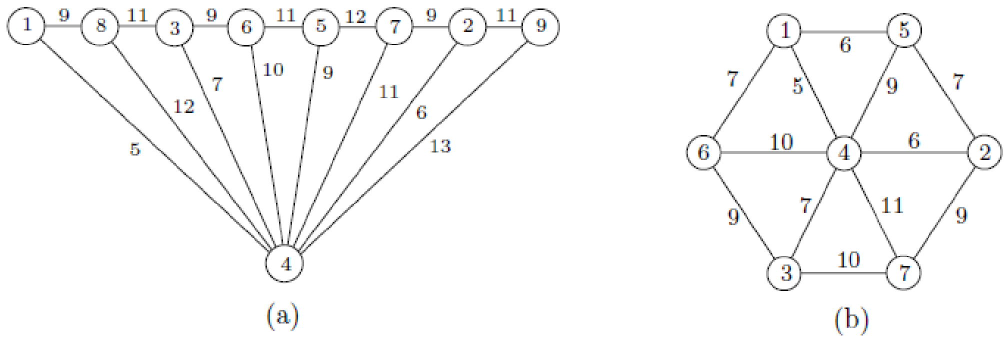

Figure 5: (a) A rainbow antimagic coloring of F_8 (b) A rainbow antimagic coloring of W_6

In the next theorem, we study the rainbow antimagic connection number of a friendship. The friendship $\mathcal{F}_n$ is a graph obtained by identifying a vertex of n copies of triangles K_3.

Theorem 6. *For* $\forall n \geq 2$ *where* $n \in \mathcal{N}$, $rac(\mathcal{F}_n) = 2n$.

Proof. Let $\mathcal{F}_n$ be a friendship with vertex set $V(\mathcal{F}_n) = \{c\} \cup \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E(\mathcal{F}_n) = \{cu_i, cv_i, u_iv_i : 1 \leq i \leq n\}$. Clearly, we have $rac(\mathcal{F}_n) \geq 2n$ according to Lemma 1. Let $f : V(\mathcal{F}_n) \to \{1, 2, \dots, 2n+1\}$ be a vertex labeling defined as follows.

$$
\begin{aligned}
f(c) &= 2,\\
f(u_1) &= 1,\\
f(u_i) &= i+1, && \text{for } i = 2,3,\dots,n,\\
f(v_i) &= 2n-i+2, && \text{for } i = 1,2,\dots,n.
\end{aligned}
$$

For the edge weights, we have

$$\begin{aligned} w_f(cu_1) &= 3, & \\ w_f(cu_i) &= i+3, & \text{for } i = 2,3,\dots,n, \\ w_f(cv_i) &= 2n-i+4, & \text{for } i = 1,2,\dots,n, \\ w_f(u_1v_1) &= 2n+2, & \\ w_f(u_iv_i) &= 2n+3, & \text{for } i = 2,3,\dots,n. \end{aligned}$$

It is clear that f induces an edge $2n$-coloring of $\mathcal{F}_n$. Since $diam(\mathcal{F}_n) = 2$, from Theorem 1 and Corollary 1, there exists a rainbow $x - y$ path for any two vertices $x, y \in V(\mathcal{F}_n)$. So, we get $rac(\mathcal{F}_n) \leq 2n$. Combining with lower bound, we can conclude that $rac(\mathcal{F}_n) = 2n$. □

For an illustration, a rainbow antimagic coloring of $\mathcal{F}_4$ is depicted in Figure 6(a).

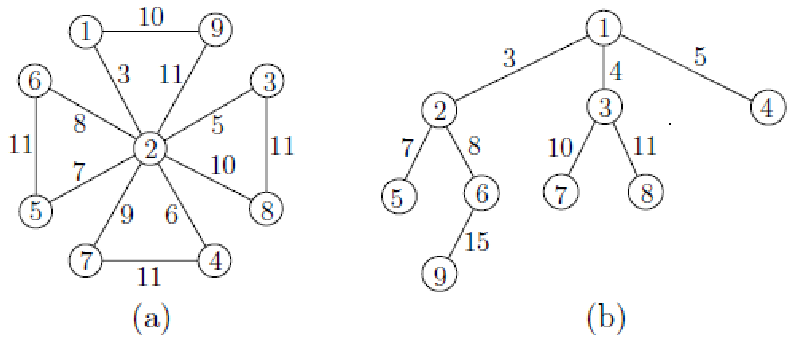

Figure 6: (a) A rainbow antimagic coloring of $\mathcal{F}_4$ (b) A rainbow antimagic coloring of T_9.

A *rooted tree* is a tree in which one vertex has been designated as the root. Suppose that T is a tree rooted at a vertex u. For a vertex $v \in V(T)\backslash\{u\}$, the *level* of v is defined as the length of the unique path from u to v. The maximum of the levels of vertices in T is called the *height* of T. In the next theorem, we give the exact value of rainbow antimagic connection number of trees.

Theorem 7. *If T_n is any tree of order $n \geq 2$ then $rac(T_n) = n - 1$.*

Proof. From Proposition 2 and Lemma 1, we have $rac(T_n) \geq n - 1$. Next, we will show that $rac(T_n) \leq n - 1$. First, without loss of generality, suppose that T_n is a rooted tree. Let $f : V(G) \to \{1, 2, \dots, n\}$ be a vertex labeling of T_n defined by applying the following algorithm.

1. Choose one vertex u in T_n arbitrarily as the root and then, define $f(u) = 1$.

2. Partition the other vertices in $V(T_n)\backslash\{u\}$ into $V_1, V_2, \dots, V_{height-1}$ where $V_i = \{v_{ij} | j = 1, 2, \dots, |V_i|\}$ is the ordered vertex set (say from left most to right most) in the i^{th}-level of T_n.

3. For $i = 1, 2, \dots, height - 1$ and $j = 1, 2, \dots, |V_i|$, define

$$f(v_{ij}) = \sum_{t=1}^{i-1} |V_t| + j + 1.$$

   Note that if $i = 1$, then $\sum_{t=1}^{i=0} |V_t| = 0$.

From the labeling above, one can verify that f induces an edge $(n-1)$-coloring of T_n. Since T_n has $n - 1$ edges, then there exists a unique rainbow $x - y$ path between every two distinct vertices $x, y \in V(T_n)$. This concludes the proof. □

For an illustration, in Figure 6(b), it is shown an example of a rainbow antimagic coloring of T_9.

In the following theorems, we characterize all graphs with rainbow antimagic connection number one or two. This characterization gives a hint for future research activities in characterizing any graphs for their rainbow antimagic connection number.

Theorem 8. *Let G be a connected graph. The $rac(G) = 1$ if and only if G is a complete graph of order two.*

Proof. Let $rac(G) = 1$. It is clear that $diam(G) = \Delta(G) = 1$. Hence, the graph G must be a complete graph of order two. Conversely, it follows from Theorem 2 that $rac(K_2) = 1$. □

Theorem 9. *Let G be a connected graph. The $rac(G) = 2$ if and only if G is a path of order three.*

Proof. Suppose that G is a path of order three. By Theorem 7, $rac(P_3) = 2$. Suppose that $rac(G) = 2$. G must have $diam(G) \leq 2$ and $\Delta(G) \leq 2$. If $diam(G) = 1$ and $\Delta(G) \leq 2$, then G is a complete graph of order at most three, which is impossible, since for $n = 2, 3$ we have $rac(K_n) \neq 2$. Thus, G has $diam(G) = 2$ and $\Delta(G) = 2$. The possible graphs are $G = C_4$ or $G = P_3$. However, according to Theorem 3, we can not choose $G = C_4$, since $rac(C_4) = 3$. Hence, $G = P_3$. □

3 Conclusion

In this paper, we have continued to initiate study rainbow antimagic connection number of graphs. We proved that any connected graph with diameter at most two admits a rainbow antimagic coloring. A general lower bound of rainbow antimagic connection number for any connected graph was obtained. The sharpness of the lower bound was proved for trees, friendships, and some cases of cycles, fans and wheels. Moreover, all graphs with the rainbow antimagic connection number of one or two are also characterized in this paper. However, apart from this results, the study of rainbow antimagic coloring are widely challenging. Thus we propose the following open problems.

Open Problem 1. *If G admits a rainbow antimagic coloring, can we develop a construction for determining the lowest rainbow antimagic connection number?*

Open Problem 2. *Let G be a connected graph. Determine the sharpest upper bound of the rainbow antimagic connection number $rac(G)$.*

4 Acknowledgement

We would like to thanks the CGANT Research Group members of the University of Jember, Indonesia 2021, on their support and motivation in finishing this paper.

References

Al Jabbar, Z.L., Dafik, Adawiyah, R., Albirri, E.R. & Agustin, I.H. (2020). On rainbow antimagic coloring of some special graph. *Journal of Physics: Conference Series*, *1465*, 012030.

Budi, H.S., Dafik, Tirta, I.M., Agustin, I.H. & Kristiana A.I. (2020). On rainbow antimagic coloring of graphs. *Journal of Physics: Conference Series*, *1832*, 012016.

Caro, Y., Lev, A., Roditty, Y., Tuza, Z. & Yuster, R. (2008). On rainbow connection. *Electron. J. Combin.*, *15*, #R57.

Chartrand, G., Johns, G.L., McKeon, K.A. & Zhang, P. (2008). Rainbow connection in graphs. *Math. Bohem.*, *133*, 85-98.

Chartrand, G. & Zhang, P. (2016). *Graphs and digraphs, Sixth ed.*., Taylor & Francis Group, 2016.

Gallian, J.A. (2020). A dynamic survey of graph labeling. *Electron. J. Combin.*, *23*, #DS6.

Hartsfield, N. & Ringel, G. (1990). *Pearls in Graph Theory.* New York: Academic Press.

Kemnitz, A. & Schiermeyer, I. (2011). Graphs with rainbow connection number two. *Discuss. Math. Graph Theory*, *31*, 313-320.

Krivelevich, M. & Yuster, R. (2010). The rainbow connection of a graph is (at most) reciprocal to its minimum degree. *J. Graph Theory*, *63*, 185-191.

Li, H., Li, X. & Liu, S. (2012). Rainbow connection of graphs with diameter 2. *Discrete Math.*, *312*, 1453-1457.

Li, H., Li, X. & Sun, Y. (2017). Rainbow connection number of graphs with diameter 3. *Discuss. Math. Graph Theory*, *37*, 141-154.

Li, X. & Shi, Y. (2013). Rainbow connection in 3-connected graphs. *Graphs Combin*, *29*, 1471-1475.

Li, X. & Sun, Y. (2017). An updated survey on rainbow connections of graphs - a dynamic survey. *Theory Appl. Graphs*, *0*(1) Article 3.

Septory, B.J., Utoyo, M.I., Dafik, Sulistiyono, B. & Agustin, I.H. (2021). On rainbow antimagic coloring of special graphs. *Journal of Physics: Conference Series*, *1836*, 012016.

Simanjuntak, R., Bertault, F. & Miller, M. (2000). Two new (a, d)-antimagic graph labelings. *Proc. of Eleventh Australasian Workshop on Combinatorial Algorithms*, *11*, 179-189.

Sulistiyono, B., Slamin, Dafik, Agustin, I.H. & Alfarisi, R. (2020). On rainbow antimagic coloring of some graphs. *Journal of Physics: Conference Series*, *1465*, 012029.